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A concise algebraic method for assessing strain in distributions of linear objects

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Abstract—Recent work has addressed the problem of determining the strain in a group of deformed linear objects. A new numerical method is described here which highlights some relationships with previous approaches and shows how analyses can be performed in three dimensions.

INTRODUCTION

OBJECTS which were initially present in deformed rocks may be used to estimate strain. Usually populations, rather than single objects, are used to reduce errors inherent in the strain estimation process. Much work has centred on the use of linear markers such as andalusite crystals (Sanderson & Meneilly 1981) and belemnites (Beach 1979, Ferguson & Lloyd 1984), and on different methods of analysis of such markers (Panozzo 1984, Sanderson & Phillips 1987). The aim of this contribution is to give a method for determining the strain in a population of deformed linear objects whose final lengths and orientations may be measured. It assumes that the objects deformed homogeneously with their surroundings. More complex methods (e.g. Ferguson & Lloyd 1984) must be used if objects and matrix deformed heterogeneously. The advantage of the method described here is that it involves no more computation than any other numeric technique for the homogeneous strain situation, yet is equally applicable in two or three dimensions. It also shows how strain of linear objects may be treated by tensor algebraic methods in the same way as may populations of lines of unspecified lengths, and populations of planes (Woodcock 1977, Harvey & Laxton 1980). In the first part of the paper the derivation of the method is described, and in the second part a worked example is given showing how it is used in practice. In the final two parts the method is compared with those of Panozzo (1984) and Sanderson & Phillips (1987) with the aim of highlighting how these different approaches to the problem are related.

DERIVATION OF THE METHOD

The derivation of the method is identical in two and three dimensions. It is closely related to ways of deducing strain from sets of vectors of unit length (e.g. Harvey & Laxton 1980) and to statistical expressions relating to orientation data (e.g. Mardia 1972, p. 223, Woodcock 1977). The principal feature of the method given here is that it uses the moment of inertia, not of a set of unit vectors, but of a set of vectors of known lengths. Let us denote the length and orientation of a linear object by vector $\mathbf{v}$. In two dimensions, if $\mathbf{v}$ has length L and orientation α then

$$\mathbf{v} = (L\cos\alpha,\ L\sin\alpha). \tag{1}$$

If this deforms homogeneously in two or three dimensions, then

$$\mathbf{v}_\mathrm{f} = \mathbf{D}\mathbf{v}_\mathrm{i},$$

where $\mathbf{D}$ is the deformation tensor. If we define a symmetric tensor

$$\mathbf{Q} = \mathbf{v}\mathbf{v} \tag{2}$$

then

$$Q_{xx} = v_x v_x$$
$$Q_{xy} = v_x v_y, \text{ etc.}$$

So, for example, in two dimensions we have

$$Q_{xx} = L^2\cos^2\alpha.$$

In both two and three dimensions $\mathbf{Q}$ changes during deformation according to:

$$\mathbf{Q}_\mathrm{f} = \mathbf{v}_\mathrm{f}\mathbf{v}_\mathrm{f} = (\mathbf{D}\mathbf{v}_\mathrm{i})(\mathbf{D}\mathbf{v}_\mathrm{i}) = \mathbf{D}(\mathbf{v}_\mathrm{i}\mathbf{v}_\mathrm{i})\mathbf{D}^\mathrm{T} = \mathbf{D}\mathbf{Q}_\mathrm{i}\mathbf{D}^\mathrm{T}. \tag{3}$$

Because this relation is linear we may take the average of both sides of this expressions:

$$\mathrm{av}(\mathbf{Q}_\mathrm{f}) = \mathrm{av}(\mathbf{D}\mathbf{Q}_\mathrm{i}\mathbf{D}^\mathrm{T}) = \mathbf{D}\,\mathrm{av}(\mathbf{Q}_\mathrm{i})\mathbf{D}^\mathrm{T}$$

and so, defining an average tensor

$$\mathbf{S} = \mathrm{av}(\mathbf{Q}),$$

we find

$$\mathbf{S}_\mathrm{f} = \mathbf{D}\mathbf{S}_\mathrm{i}\mathbf{D}^\mathrm{T}. \tag{4}$$

$\mathbf{S}_\mathrm{i}$ summarizes the initial lengths and orientations of all the lines in the distribution. It is a symmetric second rank tensor. If the initial distribution was isotropic then it should be a multiple of the unit tensor (otherwise, the maximum eigenvalue of $\mathbf{S}_\mathrm{i}$ would define a preferred initial orientation). So put

$$S_i = kg, \quad (5)$$

where k is a constant and **g** is the unit tensor. This is true for any uniform initial distribution. So

$$\mathbf{S}_f = \mathbf{D}(k\mathbf{g})\mathbf{D}^T = k\mathbf{D}\mathbf{D}^T = k\mathbf{F}, \quad (6)$$

where **F** is the Finger tensor embodying the orientation and magnitude of the strain. Although k is not known, the eigenvalues of this tensor will be proportional to the squares of the principal strains, and so the strain ratio and orientation can be determined by standard methods.

APPLICATION OF THE METHOD

It is assumed that each line is described by its length L and its orientation α clockwise from some datum direction. The line may be regarded as a vector but it is important to note that the vector has an ambiguous direction—its orientation may be specified by α or $\alpha + 180°$ with equal validity. For each line calculate the three quantities (which are components of a symmetric tensor)

$$Q_{xx} = L^2 \cos^2 \alpha$$
$$Q_{xy} = L^2 \cos \alpha \sin \alpha$$
$$Q_{yy} = L^2 \sin^2 \alpha. \quad (7)$$

Note that the ambiguity in α has no effect on these values. Now calculate the average of each of these three values for all the lines by summing them for each deformed object and dividing the result by the number of objects. These three averages are themselves components of a symmetric tensor **S**. If N is the number of lines then

$$S_{xx} = (1/N)\,\Sigma\, Q_{xx}, \text{ etc.}, \quad (8)$$

(compare Woodcock 1977). This tensor is proportional to the tensor describing the strain ellipse, as is shown above. **S** has the dimensions of area. The axial ratio R and orientation θ of the strain ellipse are extracted by the following formulae:

$$\tan 2\theta = 2S_{xy}/(S_{xx} - S_{yy}) \quad (9)$$
$$R = h + (h^2 - 1)^{1/2}, \quad (10)$$

where

$$h = (1/2)(S_{xx} + S_{yy})(S_{xx}S_{yy} - S_{xy}{}^2)^{-1/2}.$$

This expression for R is equivalent to that of Ramsay & Huber (1983, equation B.20, p. 287), noting that **S** is proportional to the Finger tensor, which in turn depends on the deformation tensor via equation (6).

To illustrate the method, Table 1 shows a worked example using only six deformed linear objects for clarity. The distribution was obtained by modelling the effect of strain on lines initially uniformly distributed at angular intervals of 30°. Only six objects are used so that the reader may check the calculation, but more data should be used when analysing real situations. The table gives the three components of the average tensor, from which are derived $\theta = 45°$ and $R = 2$ using equations (9) and (10).

Table 1. Example calculation using a model deformed distribution

Marker	L	α	$L^2 \cos^2 \alpha$	$L^2 \cos \alpha \sin \alpha$	$L^2 \sin^2 \alpha$
1	3.16	18	9.032	2.935	0.954
2	3.9	37	9.701	7.311	5.509
3	3.9	53	5.509	7.311	9.701
4	3.16	72	0.954	2.935	9.032
5	2.19	107	0.410	−1.341	4.386
6	2.19	163	4.386	−1.341	0.410
Average			$S_{xx} = 4.999$	$S_{xy} = 2.968$	$S_{yy} = 4.999$

The only assumptions here are that the strain was imposed homogeneously and that the distribution has an isotropic initial tensor **S**. All distributions that were initially uniform satisfy this criterion. Other initial distributions such as ones with, for example, four-fold symmetry, will also satisfy the criterion but may not be uniform (Wheeler 1988). Therefore it is important to test the uniformity hypothesis by unstraining the distribution according to the estimated strain and examining the derived initial distribution for uniformity. This follows the procedure of Wheeler (1984) to analyse strained elliptical objects, in which an algebraically derived strain estimate is used to unstrain the final distribution of objects. The initial distribution is then displayed in graphic form to highlight any departure from uniformity. In the case of linear markers, a histogram could be used to display the initial orientation pattern (e.g. Beach 1979).

It is important to note that this methodology applies equally to the three-dimensional case in which the lengths and orientations of lines have been measured. This is analogous to the three-dimensional method of Sanderson & Meneilly (1981). In the three-dimensional case, each line can be described by a vector **v** with three components. The tensor **Q** now has six components: xx, xy, yy, yz, zz and xz. For each linear object we have

$$Q_{xx} = v_x v_x, \text{ etc.}$$

These quantities **Q** are averaged to determine the tensor **S**, whose eigenvectors and eigenvalues are found by standard techniques.

This section has given the information necessary to use the method. The next two sections concern its relation to other procedures for analysing strain in linear objects.

RELATIONSHIP TO THE METHOD OF PANOZZO (1984)

In this method, deformed linear objects are represented by vectors **v**. It is possible to project these vectors onto a reference line of some arbitrary orientation to give a projected length P. If **n** is a unit vector parallel to the projection line, then the projected length is

$$P = \mathbf{v} \cdot \mathbf{n}. \quad (11)$$

The next step is to evaluate P for each object and then obtain the average. However, there is an ambiguity in the sign of P because there is an ambiguity in the sense in which the vector points. To avoid this, Panozzo suggests taking the positive magnitude, $|P|$, for each object. These are summed to obtain a positive value.

This whole process is repeated for different orientations of reference line, to obtain various values of $\Sigma\,|P|$. When displayed graphically, maxima and minima in the values of $\Sigma\,|P|$ are used to deduce axial ratio and orientation of the strain ellipse. If many different reference lines are used to produce a refined graph, the amount of calculation in this method is large, involving many steps of projecting and averaging.

Let us consider a minor modification to the method, in which P^2 is used instead of $|P|$. Again, this is always positive, regardless of which choice is made for the 'head' of the vector representing each linear object. P^2 may be calculated, averaged and plotted for various reference lines exactly as described by Panozzo for $|P|$. So this method is no faster, but just as valid for deducing strain. However, using P^2 we can short-cut most of the computation, as will be shown. We have

$$P^2 = (\mathbf{v}\cdot\mathbf{n})^2 = (\mathbf{n}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{n}) = \mathbf{n}\cdot(\mathbf{Qn}), \qquad (12)$$

where $\mathbf{Q}$ is the tensor defined above. For an averaging procedure on a particular reference line, $\mathbf{n}$ is the same for each object, and can be taken outside the average:

$$\begin{aligned}\mathrm{av}(P^2) &= \mathrm{av}(\mathbf{n}\cdot(\mathbf{Qn}))\\ &= \mathbf{n}\cdot(\mathrm{av}(\mathbf{Q}))\mathbf{n}\\ &= \mathbf{n}\cdot(\mathbf{Sn}).\end{aligned} \qquad (13)$$

In this expression, $\mathbf{n}$ describes the reference line, whilst $\mathbf{S}$ is a function only of the distribution. Now let the reference line have an orientation β so that $\mathbf{n}$ has components $\cos\beta$, and $\sin\beta$, and

$$\mathrm{av}(P^2) = S_{xx}\cos^2\beta + S_{xy}\cos\beta\sin\beta + S_{yy}\sin^2\beta. \qquad (14)$$

This means that the plot of $\mathrm{av}(P^2)$ against β will not only show the single maximum and minimum which are used to deduce strain, but also will be precisely sinusoidal, governed only by the three parameters which define $\mathbf{S}$. This is true whatever the number of objects being used in the analysis. There is hence no need to actually plot the graph of $\mathrm{av}(P^2)$ vs β: instead all that is required is to calculate $\mathbf{S}$, and determine its eigenvalues. This is the method already presented in this paper. In short, a minor modification to the graphical method of Panozzo (1984) shows that it illustrates the geometric meaning of the algebraic technique given here, but that the graph is not necessary to derive the strain estimate.

RELATIONSHIP TO THE METHOD OF SANDERSON & PHILLIPS (1987)

These authors propose the use of a vector average: each linear object is represented by a vector at angle 2α to some datum and with magnitude L (the length of the object). The average of these vectors has an orientation which is used to estimate the long axis of the strain ellipse and a magnitude which is related to the strain ratio. The form of this relation was derived numerically by computer simulations of a uniform distribution.

Consider a variation on the 'length-weighted' method in which the *squares* of the object lengths are used to weight the vectors. Then each linear object is represented by a vector $\mathbf{t}$, with

$$\mathbf{t} = (L^2\cos 2\alpha,\ L^2\sin 2\alpha). \qquad (15)$$

The average value of the vector $\mathbf{t}$ is a vector whose orientation yields the estimate of the strain axis, and whose normalized magnitude

$$M = |\mathrm{av}(t)|/\mathrm{av}(L^2) \qquad (16)$$

should be related to the strain ratio. So far the arguments have exactly followed the 'length weighted' method, and could determine the relation between R and M by simulations, following the example of Sanderson & Phillips. This is not necessary: an algebraic relation exists, as will now be shown. The relation is derived by noting that the vector $\mathbf{t}$ is related to the tensor $\mathbf{Q}$, referred to in previous sections, in a linear fashion,

$$\mathbf{t} = (Q_{xx} - Q_{yy},\ 2Q_{xy}). \qquad (17)$$

Similarly L^2 is another linear function, the trace of $\mathbf{Q}$:

$$L^2 = Q_{xx} + Q_{yy}. \qquad (18)$$

Since both these relations are linear, and $\mathbf{S}$ is the average of $\mathbf{Q}$, we find

$$\begin{aligned}\mathrm{av}(\mathbf{t}) &= (S_{xx} - S_{yy},\ 2S_{xy})\\ \mathrm{av}(L^2) &= S_{xx} + S_{yy}.\end{aligned} \qquad (19)$$

If the angle of $\mathrm{av}(\mathbf{t})$ to datum is 2θ, where θ is the orientation of the long axis of the strain ellipse, then

$$\tan 2\theta = 2S_{xy}/(S_{xx} - S_{yy})$$

which is precisely the formula given by the new method (equation 9). To derive the strain estimate from the 'normalized magnitude' M, note that

$$\begin{aligned}|\mathrm{av}(\mathbf{t})|^2 &= (S_{xx} - S_{xy})^2 + 4S_{xy}{}^2\\ &= (S_{xx} + S_{xy})^2 + 4S_{xy}{}^2 - 4S_{xx}S_{yy}\\ &= (\mathrm{tr}\,\mathbf{S})^2 - 4\det\mathbf{S}.\end{aligned} \qquad (20)$$

Here, $\mathrm{tr}\,\mathbf{S}$ and $\det\mathbf{S}$ are the trace and determinant of $\mathbf{S}$. So

$$M^2 = 1 - 4\det\mathbf{S}/(\mathrm{tr}\,\mathbf{S})^2.$$

If the initial distribution was uniform then $\mathbf{S}$ is proportional to the shape tensor $\mathbf{N}$ (Wheeler 1984, equation A7). From this expression is derived

$$\det\mathbf{S}/(\mathrm{tr}\,\mathbf{S})^2 = 1/(R + 1/R)^2$$

from which is found

$$M = \frac{R^2 - 1}{R^2 + 1}. \qquad (21)$$

This relation is similar in form to that shown in table 1 of Sanderson & Phillips, and is proven to apply to any initially uniform distribution. It also can be shown to be identical to the relation (4) between strain ratio and the tensor S.

To summarize this argument, it has been shown that if the 'length-weighting method' is modified to use 'length-squared weighting' then it becomes mathematically identical to the new method proposed here. The 'normalized resultant length' referred to by Sanderson & Phillips (analogous to the quantity referred to here by M) need not be calibrated against strain ratio numerically—instead equation (21) provides the relationship, and is in turn equivalent to expression (10) as described earlier.

CONCLUSIONS

The new method has the following features.

(1) The methods of Panozzo (1984) and Sanderson & Phillips (1987) do not have obvious three-dimensional generalizations. The new method applies equally well in two and three dimensions.

(2) The amount of computation required is small.

(3) The algebra involved in this method is more straightforward than the complicated integrations required in both two and three dimensions by Harvey & Laxton (1980).

(4) No numerical calibration of the method is required.

Use of this method does not prove that the assumptions behind it (namely that the distribution was uniform and the strain was homogeneous) were correct. However if the assumptions are correct then the estimate is valid. The best way to test the validity is to create the predicted unstrained distribution from the data and the strain estimate. This can be examined, visually in an appropriate form, or statistically, for uniformity (cf. Wheeler 1984). If the predicted unstrained distribution is not uniform then no other estimate of homogeneous strain could possibly produce uniformity, and the nature of the distribution and strain mechanisms must be reassessed.

Finally, the success of the tensor-algebraic approach in understanding the strain of distributions of linear objects underlines its importance in strain analysis.

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